The third approximation is

$$x_3 = \frac{a+b}{2} = \frac{2.5 + 2.75}{2} = 2.625$$

Now, $f(2.625) = (2.625)\log_{10}(2.625) - 1.2 = -0.10 < 0.$

Thus the root lies between $a = 2.625$ and $b = 2.75$.

The fourth approximation is

$$x_4 = \frac{2.625 + 2.75}{2} = 2.6875$$

The approximate value of the root of the given equation is 2.6875.

Example 4. Find positive root of the equation $x - \cos x = 0$ by using bisection method.

Solution : Let $f(x) = x - \cos x$.

Now, $f(0.5) = 0.5 - \cos 0.5 = 0.5 - 0.8775 < 0$

and $f(1) = 1 - \cos 1 = 1 - 0.5403 > 0$

Thus the root lies between 0.5 and 1.

Let $a = 0.5$ and $b = 1$

Now the first approximation is

$$x_1 = \frac{a+b}{2} = \frac{0.5+1}{2} = 0.75$$

Now, $f(x_1) = 0.75 - \cos(0.75) = 0.75 - 0.7316 > 0.$

Thus the root lies between $a = 0.5$ and $b = 0.75$.

Further calculations are shown in the following table.

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1	0.5	1	07.5	0.01831
2	0.5	0.75	0.625	- 0.18596
3	0.625	0.75	0.6875	- 0.0853
4	0.6875	0.75	0.71875	- 0.03387
5	0.71875	0.75	0.73438	- 0.00786
6	0.73438	0.75	0.7422	0.00519
7	0.73438	0.7422	0.7382	- 0.00133
8	0.7382	0.7422	0.7402	0.001866
9	0.7382	0.7402	0.73925	0.00028

Thus the approximate root is 0.7393, correct to 4 places of decimals.

EXERCISES

1. Find a root of the following equations using the bisection method

(i) $x^3 - 4x - 9 = 0$ (ii) $x^3 - 9x + 1 = 0$ (iii) $x^3 + x^2 - 1 = 0$

(iv) $x^3 - 5x + 1 = 0$ (in 5 stages) (v) $e^x = 3x$ (vi) $3x = \cos x + 1$

2. Find the negative root of $x^3 - 4x + 9 = 0$ by bisection method after five iterations. [Hint : find the positive root of $f(-x) = 0$]

ANSWERS

1. (i) 2.7065 (ii) 2.9429 (iii) 0.7549 (iv) 0.2031 (v) 0.6190 (vi)0.66664
2. −2.7065.

1.5 Newton – Raphson Method

Let x_0 be an approximate value of the root of the equation $f(x) = 0$. Let $x_1 = x_0 + h$ be the exact root of the equation, where h is very small. Then

$$f(x_1) = f(x_0 + h) = 0.$$

Expanding $f(x_0 + h)$ by Taylor's series, we have

$$f(x_0 + h) = f(x_0) + h.f'(x_0) + \frac{h^2}{2!}f''(x_0) + \ldots\ldots\ldots\ldots$$

As h is very small, neglecting the terms containing h^2, h^3 etc, we get

$$0 = f(x_0) + hf'(x_0) \quad \Rightarrow \quad h = -\frac{f(x_0)}{f'(x_0)}$$

Thus the closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \text{ provided } f'(x_0) \neq 0.$$

Now, proceeding with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \text{ provided } f'(x_1) \neq 0.$$

In general the n^{th} order approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This formula is known as **Newton – Raphson formula** or **Newton iteration formula.**

We shall mention below the criterion for the Newton – Raphson iteration to converge a root of the equation $f(x) = 0$.

The Newton – Raphson iteration converges to a root of the equation $f(x) = 0$, if

$$|f(x).f''(x)| < |f'(x)|^2$$

is true in the interval which contains the root α of $f(x) = 0$ and its initial approximation x_0

Note : Newton's method is useful in cases of large values of $f'(x_1)$. In other words when the graph of $f(x)$ while crossing the x – axis, is nearly vertical.

Example 1. Find cube root of 24, correct to three places of decimal by Newton – Raphson method.

Solution : Finding the cube root of 24 is same as solving the equation $x^3 - 24 = 0$.

Let $\quad f(x) = x^3 - 24 \Rightarrow f'(x) = 3x^2$

Now we have Newton – Raphson formula as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n^3 - 24}{3x_n^2}$$

$$\Rightarrow \quad x_{n+1} = \frac{2x_n^3 + 24}{3x_n^2} \quad \text{...... (1)}$$

Now, for $n = 0$, $x_0 = 2.8$, we have

$$x_1 = \frac{2(2.8)^3 + 24}{3(2.8)^2} = 2.887.$$

For second approximation, put $n = 1$, $x_1 = 2.887$ in (1), we get

$$x_2 = \frac{2(2.887)^3 + 24}{3(2.887)^2} = 2.885.$$

For third approximation, put $n = 2$, $x_2 = 2.8845$ in (1), we get

$$x_3 = \frac{2(2.885)^3 + 24}{3(2.885)^2} = 2.8844.$$

Thus the cube root of 24 is 2.884.

Example 2. Find the real root of $x^3 - 3x + 1 = 0$ laying between 1 and 2 upto three decimal places by Newton – Raphson method.

Solution : Let $f(x) = x^3 - 3x + 1$

Clearly $\quad f(1) = -1 < 0$ and $f(2) = 3 > 0$

Thus a root lies between 1 and 2.

Now, $\quad f'(x) = 3x^2 - 3.$ Also $f'(1) = 0.$

Thus Newton's formula cannot be applied by taking $x_0 = 1$.

Let us take $\quad x_0 = 1.5.$

Now, $\quad f(x_0) = f(1.5) = (1.5)^3 - 3(1.5) + 1 = -0.125$

$$f'(x_0) = f'(1.5) = 3(1.5)^2 - 3 = 3.75$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \left(\frac{-0.125}{3.75}\right) = 1.5 + 0.0333 = 1.5333$$

Now, $f(x_1) = f(1.5333) = (1.5333)^3 - 3(1.5333) + 1 = 0.0049$

$f'(x_1) = f'(1.5333) = 3(1.5333)^2 - 3 = 4.053$

Hence $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_2)} = 1.5333 - \left(\dfrac{0.0049}{4.053}\right) = 1.5321$

Now, $f(x_2) = f(1.5321) = (1.5321)^3 - 3(1.5321) + 1 = 0$

Hence $x_2 = 1.532$ is the required root upto three decimal places.

Example 3. Using Newton – Raphson method, find the root near 2.9 of the equation $x + \log_{10} x = 3.375$, correct to four significant figures.

Solution : Let $f(x) = x + \log_{10} x - 3.375$

$$f'(x) = 1 + \frac{1}{x}\log_{10} e$$

Now we have Newton – Raphson formula as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n + \log_{10} x_n - 3.375}{\left(1 + \frac{1}{x_n}\log_{10} e\right)} \quad \text{...... (1)}$$

Now, for $n = 0$, $x_0 = 2.9$, we have

$$x_1 = 2.9 - \frac{2.9 + \log_{10}(2.9) - 3.375}{\left(1 + \frac{1}{2.9}\log_{10} e\right)}$$

$$\Rightarrow \quad x_1 = 2.9 - \frac{0.0126}{1.1497} = 2.9109.$$

For second approximation, put $n = 1$, $x_1 = 2.9109$ in (1), we get

$$x_2 = 2.9109 - \frac{2.9109 + \log_{10}(2.9109) - 3.375}{\left(1 + \frac{1}{2.9109}\log_{10} e\right)}$$

$$\Rightarrow \quad x_2 = 2.9109 + \frac{0.0001}{1.1492} = 2.91099.$$

Thus the desired root is 2.911.

Example 4. Find the real root of $xe^x - 2 = 0$ correct to three places of decimals using Newton – Raphson method.

Solution : Let $f(x) = xe^x - 2$

Now, $f(0) = -2 < 0$ and $f(1) = e - 2 = 2.7183 - 2 = 0.7183 > 0$

Thus the root lies between 0 and 1.

Now, $f'(x) = xe^x + e^x = e^x(x + 1)$

Now, $|f(0)| > |f(1)|$. Thus we shall take the initial approximation $x_0 = 1$.

The successive approximations are shown in the following table.

n	x_n	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	$x_0 = 1$	$x_1 = 0.8679$
1	$x_1 = 0.8679$	$x_2 = 0.8528$
2	$x_2 = 0.528$	$x_3 = 0.8526$

Since x_2 and x_3 are equal upto three decimals, we have the required root as 0.853.

Example 5. Find the root of the equation $\tan x = x$, near $x = 4.5$, correct to four decimal places.

Solution : Let $f(x) = \tan x - x = 0$ be the equation.

Now $f'(x) = \sec^2 x - 1$

We have Newton – Raphson formula as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{\tan x_n - x_n}{\sec^2 x_n - 1}$$

$$\Rightarrow \quad x_{n+1} = \frac{x_n \sec^2 x_n - \tan x_n}{\sec^2 x_n - 1}$$

$$\Rightarrow \quad x_{n+1} = \frac{2x_n - \sin 2x_n}{2\sin^2 x_n}$$

Now, for first approximation, put $n = 0$, $x_0 = 4.5$, we get

$$x_1 = \frac{2(4.5) - \sin 9}{2\sin^2(4.5)} = 4.49361.$$

For second approximation, put $n = 1$, $x_1 = 4.49361$ in (1), we get

$$x_2 = \frac{2(4.49361) - \sin(8.98722)}{2\sin^2(4.49361)} = 4.49341.$$

For third approximation, put $n = 2$, $x_2 = 4.49341$ in (1), we get

$$x_3 = \frac{2(4.49341) - \sin(8.98682)}{2\sin^2(4.49341)} = 4.49341.$$

As x_2 and x_3 are identical, the required root is 4.4934, correct to four decimal places.

Example 6. Find the real root of $3x - \cos x - 1 = 0$ by Newton – Raphson method correct to 4 places.

Solution : Let $f(x) = 3x - \cos x - 1$

Now, $f(0) = -2 < 0$ and $f(1) = 3 - 0.5403 - 1 = 1.4597 > 0$

Thus the root lies between 0 and 1.

Since $|f(0)| > |f(1)|$, we shall take the initial approximation $x_0 = 1$.

The successive approximations are shown in the following table.

n	x_n	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	$x_0 = 1$	$x_1 = 0.62$
1	$x_1 = 0.62$	$x_2 = 0.6071$
2	$x_2 = 0.6071$	$x_3 = 0.6071$

Since x_2 and x_3 are equal upto four decimals, the required root is 0.6071.

Example 7. Find an iterative formula to find $\sqrt{N}$, where N is a positive number and hence find $\sqrt{5}$.

Solution : Let $x = \sqrt{N} \Rightarrow x^2 - N = 0$

Let $f(x) = x^2 - N,\ f'(x) = 2x$

We have Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n}{2} - \frac{N}{2x_n}$$

$$\Rightarrow \quad x_{n+1} = \frac{1}{2}\left(x_n - \frac{N}{x_n}\right)$$

This is the iterative formula to find $\sqrt{N}$.

Now, let $N = 5$, then

$x = \sqrt{5} \Rightarrow x$ lies between 2 and 3.

We shall take $x_0 = 2$. Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.236111$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right) = \frac{1}{2}\left(2.236111 + \frac{5}{2.236111}\right) = 2.236068$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{5}{x_3}\right) = \frac{1}{2}\left(2.236068 + \frac{5}{2.236068}\right) = 2.236068$$

Hence the value of $\sqrt{5}$ is 2.236068.

EXERCISES

1. Find a positive root of each of the following equations using Newton – Raphson method.

   (i) $x^3 + 3x - 1 = 0$ (ii) $2x^3 - 3x - 6 = 0$ (iii) $x^3 = 6x - 4$

   (iv) $x^3 - x - 2 = 0$ (v) $x^4 - x - 9 = 0$ (vi) $xe^x = \cos x$

   (vii) $x\cos x = 0$ (viii) $\log x - x + 3 = 0$ (ix) $x\sin x + \cos x = 0$

   (x) $x^4 - 3x + 1 = 0$
2. Evaluate $\sqrt{12}$ to four places of decimals by Newton – Raphson method.
3. Find the negative root of $x^3 - 2x + 5 = 0$ correct to three places of decimals by Newton – Raphson method.
4. Find the real root of $x\log_{10}x = 1.2$ correct to four decimal places.
5. Find the negative root of $x^3 - \sin x + 1 = 0$.
6. Find the positive real root of $\sqrt[3]{17}$ using Newton – Raphson method.

ANSWERS

1. (i) 0.3333 (ii) 1.78377 (iii) 0.73205 (iv) 1.521 (v) 1.813

   (vi) 0.517757 (vii) 0.739 (viii) 0.052 (ix) 2.2984 (x) 1.31
2. 3.4641 3. −2.095 4. 2.741 5. −1.249 6. 2.571

1.6 Regula – Falsi Method (Method of false position)

Consider the equation $f(x) = 0$, where $f(x)$ is a continuous function. Supposing $f(x)$ is such that $f(a)$ and $f(b)$ are of opposite signs. Then the graph of $y = f(x)$ crosses the x – axis between these two points. This shows that a root of $f(x) = 0$ lies between a and b.

Now let $A = (a, f(a))$ and $B = (b, f(b))$ be the two points on the curve $y = f(x)$. The equation of the chord joining A and B is given by

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad ..... (1)$$

In this method we replace the curve AB by means of the chord and take the point $(x_1, 0)$ of intersection of the chord with the x – axis as an the first approximation to the root, which is obtained by putting $y = 0$ and $x = x_1$ in (1). Thus we have

$$\frac{-f(a)}{x_1 - a} = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \qquad x_1 - a = -\frac{f(a)(b-a)}{f(b)-f(a)}$$

$$\Rightarrow \qquad x_1 = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

$$\Rightarrow \qquad x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} \qquad \text{..... (2)}$$

Now if $f(a)$ and $f(x_1)$ are of opposite signs, then the root lies between a and x_1. Replace b by x_1 in (2) we get the next approximation x_2.

Supposing if $f(b)$ and $f(x_1)$ are of opposite signs, then the root lies between x_1 and b. In this case we replace a by x_1 in (2) to get the second approximation.

This process is repeated till the root is obtained to desired accuracy.

The above iteration procedure based on above formula (2) is known as method of **false position** or **Regula – Falsi** method. This method is an improvement over the bisection method.

Working Rule

Step 1. Choose two numbers a and b such that $f(a)$ and $f(b)$ are of opposite signs. That is a root lies between a and b.

Step 2. Use the formula $\qquad x_1 = \dfrac{af(b)-bf(a)}{f(b)-f(a)}$

to find the first approximation to the root.

Step 3. Find $f(x_1)$ and find whether $f(a)$ and $f(x_1)$ or $f(x_1)$ and $f(b)$ are of opposite signs. Supposing $f(a)$ and $f(x_1)$ are of opposite signs, then replace b by x_1 in the above formula to get the next approximation x_2. If $f(x_1)$ and $f(b)$ are of opposite signs, then replace a by x_1 in the above formula to get x_2.

Step 4. Repeat the process till the root is found to desired accuracy.

Note : To find a root correct to n decimal places we carry out the above process till two successive approximation values coincide upto n decimal places. Further the sequence $x_1, x_2, \ldots, x_n, \ldots$ will converge to the required root. In practice, we get x_i and x_{i+1} such that $|x_i$ and $x_{i+1}| < \varepsilon$, where ε is the required accuracy.

Example 1. The equation $x^4 - x - 10 = 0$ has one root between 1.8 and 2. Find the root correct to 3 places of decimal by the method of false position.

Solution : Let $\qquad f(x) = x^4 - x - 10$.

Now $\qquad x_0 = 1.8$ and $x_1 = 2$

$\Rightarrow \qquad f(x_0) = f(1.8) = -1.3$ and $f(x_1) = f(2) = 4$

Thus $f(x_0)$ and $f(x_1)$ are of oposite signs.

The first order approximation is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{(1.8)(4) - (2)(-1.3)}{4 + 1.3} = 1.849$$

Now $f(x_2) = f(1.849) = -0.161.$

Thus $f(x_1)$ and $f(x_2)$ are of opposite signs. Now next approximation is given by

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$x_3 = \frac{(2)(-0.161) - (1.849)(4)}{-0.161 - 4} = 1.855$$

Now $f(x_3) = f(1.854) = -0.192.$

Thus $f(x_1)$ and $f(x_3)$ are of opposite signs. Now next approximation is given by

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

$$x_4 = \frac{(2)(-0.0192) - (1.855)(4)}{-0.0192 - 4} = 1.855$$

Since $x_3 = x_4 = 1.855$, the required approximated root is 1.855.

Example 2. Find the root of the equation $2x - \log_{10} x = 7$ which lies between 3.5 and 4 by Regula – Falsi method.

Solution : Let $f(x) = 2x - \log_{10} x - 7,\quad x_0 = 3.5$ and $x_1 = 4$

The first approximation is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{(3.5)(0.3979) - (4)(0.5441)}{0.3979 - 0.5441} = 3.7888$$

Now $f(x_2) = f(3.7888) = -0.0009$ and $f(x_2) = f(4) = 0.3979,$

which are of opposite signs. Thus the second approximation is given by

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$x_3 = \frac{(4)(3.7888) - (3,7888)(0.3979)}{0.37888 - 0.3979} = 3.789$$

The required root to three decimal places is 3.789.

Example 3. Find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$ upto 3 places of decimals using Regula – Falsi method.

Solution : Let $f(x) = x^3 - 3x + 1$

Clearly $f(1) = -1 < 0$ and $f(2) = 3 > 0$

Let $a = 1$ and $b = 2$. The first approximation is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{1 . f(2) - 2 . f(1)}{f(2) - f(1)} = \frac{5}{4} = 1.25$$

Now, $f(x_1) = (1.25)^3 - 3(1.25) + 1 = -0.7969 < 0$

Thus the root lies between $a = 1.25$ and $b = 2$.

The second approximation is given by

$$x_2 = \frac{(1.25) f(2) - 2 . f(1.25)}{f(2) - f(1.25)} = 1.4074$$

The successive approximations are given in the following table.

i	a	b	$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x_i)$
1	1	2	1.25	$-0.7969 < 0$
2	1.25	2	1.4074	$-0.4345 < 0$
3	1.4074	2	1.4824	$-0.1896 < 0$
4	1.4824	2	1.5132	$-0.0747 < 0$
5	1.5132	2	1.525	-0.0284
6	1.525	2	1.5295	-0.0104
7	1.5295	2	1.5311	-0.004
8	1.5311	2	1.532	-0.0016
9	1.5317	2	1.532	

Here x_8 and x_9 are equal correct to three places of decimals. Thus the required root is 1.532.

Example 4. Solve for a positive root of $x^3 - 4x + 1 = 0$ by Regula – Falsi method.

Solution : Let $f(x) = x^3 - 4x + 1$

Clearly $f(0) = 1 > 0$ and $f(1) = -2 < 0$. Also $f(2) = 1 > 0$

Thus one root lies between 0 and 1 and another root lies between 1 and 2.

We shall find the root that lies between 0 and 1.

Let $a = 0$ and $b = 1$. The first approximation is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{0 . f(1) - 1 . f(0)}{f(1) - f(0)} = 0.33333$$

Now, $f(x_1) = (0.33333)^3 - 4(0.33333) + 1 = -0.2963 < 0$

Thus the root lies between 0 and = 0.33333.

The successive approximations are given in the following table.

i	a	b	$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x_i)$
1	0	1	0.33333	−0.2963
2	0	0.33333	0.25714	−0.011558
3	0	0.25714	0.25420	−0.0003742
4	0	0.25420	0.25410	−0.0000129
5	0	0.25410	0.25410	

Here x_4 and x_5 are equal. Thus the required root is 0.25410.

Example 5. Find a positive root of $xe^x = 2$ by the method of false position.

Solution : Let $f(x) = xe^x - 2$

Now $f(0) = -2 < 0$ and $f(1) = 0.71828 > 0$.

Thus a root lies between 0 and 1.

Let $a = 0$ and $b = 1$. The first approximation is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{0 - f(0)}{f(1) - f(0)} = 0.73576$$

Now, $f(x_1) = f(0.73576) = -0.464423 < 0$

Thus the root lies between 0. 73576 and 1.

The successive approximations are given in the following table.

i	a	b	$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x_i)$
1	0	1	0.73576	−0.46442
2	0.73576	1	0.839521	−0.056293
3	0.83952	1	0.851184	−0.006171
4	0.85114	1	0.852452	−0.0006556
5	0.852452	1	0.85261	−0.000309
6	0.85261	1	0.85261	

Here $x_5 = x_6$ upto five places of decimals. Thus the required root is 0.5261.

Example 6. Find the positive root of $x^2 - \log_{10}x - 12 = 0$ by Regula – Falsi method.

Solution : Let $f(x) = x^2 - \log_{10}x - 12$

Clearly $f(3) = 9 - \log_{10}3 - 12 = -3.4771 < 0$

$f(4) = 16 - \log_{10}4 - 12 = 3.3979 > 0$

or $$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{0 . f(1.5) - (1.5)f(0)}{f(1.5) - f(0)} = \frac{-(1.5).1}{-1.25 - 1} = 0.7059$$

Now, $f(x_2) = f(0.7059) = 0.4592$

The next approximation is

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1}$$

$$= \frac{(1.5)(0.4592) - (0.7059)(-1.125)}{0.4592 - (-1.125)} = 0.9361$$

Now, $f(0.9361) = 0.0677$

The next approximation is

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{x_2 f_3 - x_3 f_2}{f_3 - f_2}$$

$$= \frac{(0.7059)(0.0677) - (0.9361)(0.4592)}{0.9361 - 0.4592} = 0.9757$$

Now, $f(0.9757) = 0.0484$

The next approximation is

$$x_5 = \frac{x_3 f_4 - x_4 f_3}{f_4 - f_3} = \frac{(0.9361)(0.0484) - (0.9757)(0.0677)}{0.0484 - 0.0677} = 1.0777$$

Now, $f(1.0777) = -0.1550$

The next approximation is

$$x_6 = \frac{x_4 f_5 - x_5 f_4}{f_5 - f_4} = \frac{(0.9757)(-0.1550) - (0.0777)(0.0484)}{-0.1550 - 0.0484} = 0.99997.$$

Thus a positive root is 1.

Example 2. Using Secant method find the root of the equation $\cos x - xe^x = 0$ that lies between 0 and 1.

Solution : Let $f(x) = \cos x - x e^x$

Clearly f(0) = 1 and f(1) = – 2.17798. Thus a root lies between 0 and 1.

Let $x_0 = 0$ and $x_1 = 1$. Thus $f_0 = 1$ and $f_1 = -2.17798$

The first approximation x_2 is given by

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{0 . (-2.17798) - 1(1)}{-2.17798 - 1} = 0.31467$$

Now, $f(x_2) = f_2 = f(0.31467) = 0.51987$

The second approximation is

$$x_3 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1}$$

$$= \frac{1\,(0.51987) - (0.31467)(-2.17798)}{0.51987 - (-2.17798)} = 0.44673$$

Now, $f(x_3) = f_3 = f(0.44673) = 0.20354$

The third approximation is given by

$$x_4 = \frac{x_2 f_3 - x_3 f_2}{f_3 - f_2}$$

$$= \frac{(0.31467)(0.20354) - (0.44673)(0.51987)}{0.20354 - 0.51987} = 0.53171$$

Now, $f(x_4) = f_4 = f(0.53171) = -0.04295$

The fourth approximation is given by

$$x_5 = \frac{x_3 f_4 - x_4 f_3}{f_4 - f_3}$$

$$= \frac{(0.44673)(-0.04295) - (0.53171)(0.20354)}{-0.04295 - 0.20354} = 0.51690$$

Now, $f(x_5) = f_5 = f(0.51690) = 0.002593$

The fifth approximation is given by

$$x_6 = \frac{x_4 f_5 - x_5 f_4}{f_5 - f_4} = 0.51775.$$

Now, $f(x_6) = f_6 = f(0.51775) = 0.00003011$

The sixth approximation is given by

$$x_7 = \frac{x_5 f_6 - x_6 f_5}{f_6 - f_5} = 0.51776.$$

Since x_6 and x_7 are identical upto fourth place of decimals, $x = 0.51776$ is a root correct to four places of decimals.

EXERCISES

Find the roots of the following equations correct to 3 places of decimals by employing secant method.

1. $x^3 - 5x + 1 = 0$ in $(0, 0.5)$
2. $x^4 - x - 10 = 0$ in $(1.5, 2.0)$
3. $x \sin x = 1$ in $(1.0, 1.5)$
4. $x\log_{10} x = 1.2$ in $(2.5, 3.0)$
5. $x = e^{-x}$ in $(0.5, 0.75)$

ANSWERS

1. 0.219 2. 1.856 3. 1.114 4. 2.741 5. 0.567

1.8 Numerical Solutions of Non – homogeneous System of Equations

The readers are familiar with the methods of solving a system of non – homogeneous system of equations by Cramer's rule and matrix method. Sometimes these methods becomes tedious, in particular for large systems. We shall see in this section few numerical methods of solving such systems out of which some are direct methods and other are iterative methods of solving.

1.9 Gauss Elimination Method

In this method we reduce the system of equations to an upper triangular system and the unknowns are determined by back sudstitution.

In other words, the given system

$$a_{11}x + a_{12}y + a_{13}z = d_1$$
$$a_{21}x + a_{22}y + a_{23}z = d_2$$
$$a_{31}x + a_{32}y + a_{33}z = d_3$$

is reduced to form

$$a_{11}x + a_{12}y + a_{13}z = d_1$$
$$a_{22}'y + a_{23}'z = d_2'$$
$$a_{33}''z = d_3''$$

and the values of x, y and z are obtained by back susstitution. That is z is obtained from 3rd equation, the y is obtained by putting the z value in the 2nd equation and the value of x from the 1st equation by putting y and z values.

This process of solving the system is called **Gauss elimination method.**

The actual process of reducing the given system to upper triangular system is explained below through the following example by writing the system in the matrix form. To this end we apply elementary row operations on the matrix.

Example 1. Solve by Gauss elimination method

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

Solution : The matrix form of the equation is of the form $A \cdot X = B$

i.e.
$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 18 \\ 16 \end{pmatrix}$$

Consider the adjugate matrix $[A : B]$

i.e.
$$\begin{pmatrix} 2 & 1 & 1 & : & 10 \\ 3 & 2 & 3 & : & 18 \\ 1 & 4 & 9 & : & 16 \end{pmatrix}$$

Step 1. Observe the coefficient of x in the first equation is not zero. Now eliminate x term from the second equation (i.e. it makes the coefficient of x in second equation as zero) by applying the row operation $R_2 \to R_2 - \frac{3}{2}R_1$ and eliminate x term from the third equation by applying $R_3 \to R_3 - \frac{1}{2}R_1$

i.e. $$\begin{pmatrix} 2 & 1 & 1 & : & 10 \\ 0 & 1/2 & 3/2 & : & 3 \\ 0 & 7/2 & 17/2 & : & 11 \end{pmatrix}$$

Here the first equation is called **Pivotal equation** and 2 (coefficient of x in first equation) is called **first pivot.**

Step 2. Observe the coefficient of y in the second equation is not zero. Now eliminate y term in the third equation (i.e. make the coefficient of y in the third equation as zero) by applying the row operation

$$R_3 \to R_3 - \frac{(7/2)}{(1/2)}R_2; \quad \text{i.e.} \quad R_3 \to R_3 - 7R_2.$$

i.e. $$\begin{pmatrix} 2 & 1 & 1 & : & 10 \\ 0 & 1/2 & 3/2 & : & 3 \\ 0 & 0 & -2 & : & -10 \end{pmatrix}$$

Here the second equation is called the pivotal equation and $\frac{1}{2}$ (coefficient of y in the 2nd equation) is the new pivot.

Now the corresponding upper triangular system is

$$2x + y + z = 10.$$
$$\frac{1}{2}y + \frac{3}{2}z = 3$$
$$-2z = -10 \quad \Rightarrow \quad z = 5.$$

The values of x, y, z are found from the above system by back substitution.

Now, $\frac{1}{2}y + \frac{3}{2}z = 3 \quad \Rightarrow \quad \frac{1}{2}y + \frac{15}{2} = 3 \quad \Rightarrow \quad y = -9$

$2x + y + z = 10 \quad \Rightarrow \quad 2x - 9 + 5 = 10 \quad \Rightarrow \quad x = 7$

Thus the solution is $x = 7, \quad y = -9$ and $z = 5$.

We shall explain the above method for general equation

Consider the system

$$a_{11}x + a_{12}y + a_{13}z = d_1$$
$$a_{21}x + a_{22}y + a_{23}z = d_2$$
$$a_{31}x + a_{32}y + a_{33}z = d_3$$

Consider the matrix form, $A \cdot X = B$

i.e. $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Now consider,

$$[A.B] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & : & d_1 \\ a_{21} & a_{22} & a_{23} & : & d_2 \\ a_{31} & a_{32} & a_{33} & : & d_3 \end{pmatrix}$$

Here $a_{11} \neq 0$ is the first pivot, make the entries below this entry to be zero by applying $R_2 \to R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1$ and $R_3 \to R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1$. Thus we get

$$[A.B] \sim = \begin{pmatrix} a_{11} & a_{12} & a_{13} & : & d_1 \\ & a'_{22} & a'_{23} & : & d_2' \\ & a'_{32} & a'_{33} & : & d_3' \end{pmatrix}$$

Here $a'_{22} \neq 0$ is the new pivot, make the entries below this entry to be zero by applying $R_3 \to R_3 - \left(\frac{a'_{32}}{a'_{22}}\right)R_2$. Thus we get

$$[A.B] \sim \begin{pmatrix} a_{11} & a_{12} & a_{13} & : & d_1 \\ 0 & a'_{22} & a'_{23} & : & d_2' \\ 0 & 0 & a_{33}'' & : & d_3'' \end{pmatrix}$$

This gives us the upper triangular system.

$$a_{11}x + a_{12}y + a_{13}z = d_1$$
$$a'_{21}y + a'_{22}z = d_2'$$
$$a_{33}''z = d_3''$$

Now the values of x, y, z are found by back substitution.

Note : The Gauss elimination method fails if any one of the pivots a_{11}, a'_{21} or a''_{33} becomes zero. In such circumstances we interchange the rows in $[A : B]$ in a proper way so that pivots are non zero.

Example 2. Solve by Gauss elimination method

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$

Solution : The given system can be written as $A.X = B$

i.e. $$\begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ -5 \\ -6 \end{pmatrix}$$

Now consider,

$$[A . B] = \begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 1 & 1 & 1 & 4 & : & -6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34/5 & 4/5 & 4/5 & : & 56/5 \\ 0 & 4/5 & 29/5 & 4/5 & : & -29/5 \\ 0 & 4/5 & 4/5 & 19/5 & : & -34/5 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_1 - 1/5\, R_1 \\ R_3 \to R_3 - 1/5\, R_1 \\ R_4 \to R_4 - 1/5\, R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 4 & : & 56 \\ 0 & 4 & 29 & 4 & : & -29 \\ 0 & 4 & 4 & 19 & : & -34 \end{pmatrix}. \qquad \begin{matrix} R_2 \to 5R_2 \\ R_3 \to 5R_3 \\ R_4 \to 5R_4. \end{matrix}$$

$$\sim \begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 4 & : & 56 \\ 0 & 0 & 970 & 120 & : & -1210 \\ 0 & 0 & 120 & 630 & : & -1380 \end{pmatrix} \qquad \begin{matrix} R_3 \to 34R_3 - 4R_2 \\ R_4 \to 34R_4 - 4R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 4 & : & 56 \\ 0 & 0 & 97 & 12 & : & -121 \\ 0 & 0 & 12 & 63 & : & -138 \end{pmatrix} \qquad \begin{matrix} R_3 \to 1/10\, R_3 \\ R_4 \to 1/10\, R_4 \end{matrix}$$

$$\sim \begin{pmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 4 & : & 56 \\ 0 & 0 & 97 & 12 & : & -121 \\ 0 & 0 & 0 & 5967 & : & 11934 \end{pmatrix} \qquad R_4 \to 97R_4 - 12R_3$$

Thus the upper triangular system is

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$34x_2 + 4x_3 + 4x_4 = 56$$
$$97x_3 + 12x_4 = -121$$
$$5967x_4 = -11934$$

Now, $5967x_4 = -11934 \Rightarrow x_4 = -2$

$97x_3 + 12x_4 = -121 \Rightarrow 97x_3 - 24 = -121$

$\Rightarrow x_3 = -1$

Similarly, 2nd equation $\Rightarrow x_2 = 2$ and 1st equation $\Rightarrow x_1 = 1$.

Thus the solution is

$$x_1 = 1,\ x_2 = 2,\ x_3 = -1 \text{ and } x_4 = -2.$$

The Gauss elimination method can be applied with minor modification. This modified method is called the method of **partial pivoting.**

In this method, in the first stage, we choose the numerically largest coefficient of x from all the equations and brought as the first pivot. This can be done by interchanging the first equation with the equation having the largest coefficient of x. Again in the second stage we choose the numerical largest coefficient of y from the remaining equations and brought as second pivot. This is also done by interchanging the equations suitably. This process is continued till we arrive at the equation with single variable.

Example 3. Solve by Gauss elimination method

$$\begin{aligned} 2x + y + 4z &= 16 \\ 3x + 2y + z &= 10 \\ x + 3y + 3z &= 16 \end{aligned}$$

Solution : For this system we have

$$[A \,.\, B] = \begin{pmatrix} 2 & 1 & 4 & : & 16 \\ 3 & 2 & 1 & : & 10 \\ 1 & 3 & 3 & : & 16 \end{pmatrix}$$

Here the second equation contains the largest element, we shall bring this to first row by applying $R_1 \sim R_2$

$$[A \,.\, B] \sim \begin{pmatrix} 3 & 2 & 1 & : & 10 \\ 2 & 1 & 4 & : & 16 \\ 1 & 3 & 3 & : & 16 \end{pmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 3 & 2 & 1 & : & 10 \\ 0 & -1 & 10 & : & 28 \\ 0 & 7 & 8 & : & 38 \end{pmatrix} \qquad \begin{matrix} R_2 \to 3R_2 - 2R_1 \\ R_3 \to 3R_3 - R_1 \end{matrix}$$

Here the second column contains the largest element, 7. We shall bring this to second row by applying $R_3 \leftrightarrow R_2$

$$\sim \begin{pmatrix} 3 & 2 & 1 & : & 10 \\ 0 & 7 & 8 & : & 38 \\ 0 & -1 & 10 & : & 28 \end{pmatrix} \qquad R_2 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 3 & 2 & 1 & : & 10 \\ 0 & 7 & 8 & : & 38 \\ 0 & 0 & 78 & : & 234 \end{pmatrix} \qquad R_3 \to 7R_3 + R_2$$

Thus the upper triangular system is

$$\begin{aligned} 3x + 2y + z &= 10 \\ 7y + 8z &= 38 \\ 78z &= 234 \end{aligned}$$

Now, $\quad 78z = 234 \quad \Rightarrow \quad z = 3$

$7y + 8z = 38 \quad \Rightarrow \quad 7y + 24 = 38 \quad \Rightarrow \quad y = 2$

and $\quad 3x + 2y + z = 10 \quad \Rightarrow \quad 3x + 4 + 3 = 10 \quad \Rightarrow \quad x = 1$

Thus the solution is, $\quad x = 1, \quad y = 2, \quad z = 3.$

EXERCISES

1. Solve the following equations by Gauss elimination method.

   (a) $x + 4y - z = -5; \quad x + y - 6z = -12; \quad 3x - y - z = 4$

(b) $x + y + z = 6$; $3x + 3y + 4z = 20$; $2x + y + 3z = 13$

(c) $2x + 2y + z = 12$; $3x + 2y + 2z = 8$; $5x + 10y - 8z = 10$

(d) $5x_1 - x_2 - 2x_3 = 142$; $x_1 - 3x_2 - x_3 = -30$; $2x_1 - x_2 - 3x_3 = 5$

(e) $2x_1 + 4x_2 + x_3 = 3$; $3x_1 + 2x_2 - 2x_3 = -2$; $x_1 - x_2 + x_3 = 6$

(f) $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$;
$x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$

2. Solve the following equations by employing the method of partial pivoting.

(a) $5x - 2y + z = 4$; $7x + y - 5z = 4$; $3x + 7y + 4z = 10$

(b) $2x - 3y + z = -1$; $x + 4y + 5z = 25$; $3x - 4y + 5z = 2$.

(c) $x_1 + x_2 - x_3 = 2$; $2x_1 + 3x_2 + 5x_3 = -3$; $3x_1 + 2x_2 - 3x_3 = 6$

(d) $x_1 - x_2 + x_3 + x_4 = 6$; $2x_1 - x_3 - x_4 = 5$;
$2x_1 - 2x_2 + x_4 = 4$; $x_2 + x_3 - x_4 = 3$

ANSWERS

1. (a) $x = \frac{117}{71}, y = -\frac{81}{71}, z = \frac{148}{71}$ (b) $x = 3, \; y = 1, \; z = 2$

(c) $x = -12.75, \; y = 14.375, \; z = 8.75$

(d) $x_1 = 39.345, \; x_1 = 16.793, \; x_1 = 18.97$

(e) $x_1 = 2, \; x_2 = -1, \; x_3 = 3$

(f) $x_1 = 2.5, \; x_2 = 14.7, \; x_3 = 0.2; \; x_4 = -23.4$

2. (a) $x = \frac{122}{109}, y = \frac{284}{327}, z = \frac{46}{327}$ (b) $x = 5.794, y = 4.3235, z = 0.3824$

(c) $x_1 = 1, \; x_2 = 0, \; x_3 = -1$

(d) $x_1 = \frac{17}{3}, \; x_2 = 6, \; x_3 = \frac{5}{3}; \; x_4 = \frac{14}{3}.$

1.10 Gauss – Jordan Method

In this method we not only eliminate the coefficients of the unknowns below the pivotal equation but also in the equations above it and there by reducing the system to diagonal form of the system – i.e. each equation involving only one unknown. From these equations we get directly the value of the unknown. This process is called **Gauss – Jordan** method.

Example 1. Applying Gauss – Jordan method to solve

$$4x_1 - x_2 = 1$$
$$-x_1 + 4x_2 - x_3 = 0$$
$$-x_2 + 4x_3 - x_4 = 0$$
$$-x_3 + 4x_4 = 0$$

Solution : The agumented matrix of the system is

$$[A\,.\,B] = \begin{pmatrix} 4 & -1 & 0 & 0 & : & 1 \\ -1 & 4 & -1 & 0 & : & 0 \\ 0 & -1 & 4 & -1 & : & 0 \\ 0 & 0 & -1 & 4 & : & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1/4 & 0 & 0 & : & 1/4 \\ -1 & 4 & -1 & 0 & : & 0 \\ 0 & -1 & 4 & -1 & : & 0 \\ 0 & 0 & -1 & 4 & : & 0 \end{pmatrix} \qquad R_1 \to \frac{1}{4}R_1$$

$$\sim \begin{pmatrix} 1 & -1/4 & 0 & 0 & : & 1/4 \\ 0 & 15/4 & -1 & 0 & : & 1/4 \\ 0 & -1 & 4 & -1 & : & 0 \\ 0 & 0 & -1 & 4 & : & 0 \end{pmatrix} \qquad R_2 \to R_2 + R_1$$

$$\sim \begin{pmatrix} 1 & -1/4 & 0 & 0 & : & 1/4 \\ 0 & 1 & -4/15 & 0 & : & 1/15 \\ 0 & -1 & 4 & -1 & : & 0 \\ 0 & 0 & -1 & 4 & : & 0 \end{pmatrix} \qquad R_2 \to \frac{4}{15}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & -1/15 & 0 & : & 4/15 \\ 0 & 1 & -4/15 & 0 & : & 1/15 \\ 0 & 0 & 56/15 & -1 & : & 1/15 \\ 0 & 0 & -1 & 4 & : & 0 \end{pmatrix} \qquad \begin{matrix} R_1 \to R_1 + \frac{1}{4}R_2 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1/15 & 0 & : & 4/15 \\ 0 & 1 & -4/15 & 0 & : & 1/15 \\ 0 & 0 & 1 & -15/56 & : & 1/56 \\ 0 & 0 & -1 & 4 & : & 0 \end{pmatrix} \qquad R_3 \to \frac{15}{56}R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1/56 & : & 15/56 \\ 0 & 1 & 0 & -1/14 & : & 1/14 \\ 0 & 0 & 1 & -15/56 & : & 1/56 \\ 0 & 0 & 0 & 209/56 & : & 1/56 \end{pmatrix} \qquad \begin{matrix} R_1 \to R_1 + 1/15 R_3 \\ R_2 \to R_2 + 4/15\, R_3 \\ R_4 \to R_4 + \quad R_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1/56 & : & 15/56 \\ 0 & 1 & 0 & -1/14 & : & 1/14 \\ 0 & 0 & 1 & -15/56 & : & 1/56 \\ 0 & 0 & 0 & 1 & : & 1/209 \end{pmatrix} \qquad R_4 \to \frac{56}{209}R_4$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & : & 56/209 \\ 0 & 1 & 0 & 0 & : & 15/209 \\ 0 & 0 & 1 & 0 & : & 4/209 \\ 0 & 0 & 0 & 1 & : & 1/209 \end{pmatrix} \qquad \begin{matrix} R_1 \to R_1 + 1/56 R_4 \\ R_2 \to R_2 + 1/14\, R_4 \\ R_3 \to R_3 + 15/56\, R_3 \end{matrix}$$

The above matrix yields,

$$x_1 = \frac{56}{209},\ x_2 = \frac{15}{209},\ x_3 = \frac{4}{209},\ x_4 = \frac{1}{209}.$$

Example 2. **Applying Gauss – Jordan method solve**

$$10x + y + z = 12$$
$$x + 10y + z = 12$$
$$x + y + 10z = 12$$

Solution : The agumented matrix of the system is

$$[A.B] = \begin{pmatrix} 10 & 1 & 1 & : & 12 \\ 1 & 10 & 1 & : & 12 \\ 1 & 1 & 10 & : & 12 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1/10 & 1/10 & : & 6/5 \\ 1 & 10 & 1 & : & 12 \\ 1 & 1 & 10 & : & 12 \end{pmatrix} \qquad R_1 \to \frac{1}{10} R_1$$

$$\sim \begin{pmatrix} 1 & 1/10 & 1/10 & : & 6/5 \\ 0 & 99/10 & 9/10 & : & 54/5 \\ 0 & 9/10 & 99/10 & : & 54/5 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1/10 & 1/10 & : & 6/5 \\ 0 & 1 & 1/11 & : & 12/11 \\ 0 & 9/10 & 99/10 & : & 54/5 \end{pmatrix} \qquad R_2 \to \frac{10}{99} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 1/11 & : & 12/11 \\ 0 & 1 & 1/11 & : & 12/11 \\ 0 & 0 & 108/11 & : & 108/11 \end{pmatrix} \qquad \begin{matrix} R_1 \to R_1 - 1/10\, R_2 \\ R_3 \to R_3 - 9/10\, R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1/11 & : & 12/11 \\ 0 & 1 & 1/11 & : & 12/11 \\ 0 & 0 & 1 & : & 1 \end{pmatrix} \qquad R_4 \to \frac{11}{108} R_4$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{pmatrix} \qquad R_1 \to R_1 - \frac{1}{11} R_4$$

This yields, $x = 1,\ y = 1$ and $z = 1.$

EXERCISES

Solve the following equations by Gauss – Jordan method

1. $2x + y + z = 10;$ $3x + 2y + 3z = 18;$ $x + 4y + 9z = 16$
2. $2x - 3y + z = -1;$ $x + 4y + 5z = 25;$ $3x - 4y + z = 2$
3. $x_1 - 2x_2 + 3x_3 = 2;$ $3x_1 - x_2 + 4x_3 = 4;$ $2x_1 + x_2 - 2x_3 = 5$

4. $x_1 + 2x_2 + x_3 = 8$; $2x_1 + 3x_2 + 4x_3 = 20$; $4x_1 + 3x_2 + 2x_3 = 16$

5. $2x - y + z = -1$; $2y - z + u = 1$; $x + 2z - u = -1$; $x + y + 2u = 3$

6. $2x_1 + x_2 + 5x_3 + x_4 = 5$; $x_1 + x_2 - 3x_3 + 4x_4 = -1$;
$3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$

ANSWERS

1. $x = 7, y = -9, z = 5$
2. $x = 8.7, y = 5.7, z = -1.3$
3. $x_1 = 2.2, x_2 = -1.4, x_3 = -1$
4. $x_1 = 1, x_2 = 2, x_3 = -3$
5. $x = -1, y = 0, z = 1, u = 2$
6. $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 0.8.$

1.11 Jacobi Iteration Method

Consider the system of equations

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned} \qquad \text{..... (1)}$$

in which the diagonal coefficients are not zero and are large as compared to other coefficients. In such case the system is called a **diagonally dominant system.**

Solving these equations for x, y, z respectively, the above system can be written as

$$\begin{aligned} x &= \frac{1}{a_1}\left[d_1 - b_1 y - c_1 z\right] \\ y &= \frac{1}{b_2}\left[d_2 - a_2 x - c_2 z\right] \\ z &= \frac{1}{c_3}\left[d_3 - a_3 x - b_3 y\right] \end{aligned} \qquad \text{..... (2)}$$

Supposing the initial approximations of the roots are x_0, y_0, z_0. Substituting these values on the right we get the first approximations x_1, y_1, z_1 or first iterates,

$$x_1 = \frac{1}{a_1}\left[d_1 - b_1 y_0 - c_1 z_0\right]$$

$$y_1 = \frac{1}{b_2}\left[d_2 - a_2 x_0 - c_2 z_0\right]$$

$$z_1 = \frac{1}{c_3}\left[d_3 - a_3 x_0 - b_3 y_0\right]$$

Substituting these on the right hand sides of (2), the second approximations are given by

$$x_2 = \frac{1}{a_1}\left[d_1 - b_1 y_1 - c_1 z_1\right]$$

$$y_2 = \frac{1}{b_2}\left[d_2 - a_2 x_1 - c_2 z_1\right]$$

$$z_2 = \frac{1}{c_3}\left[d_3 - a_3 x_1 - b_3 y_1 \right]$$

This process is repeated till we get the desired order of approximation of roots or the difference between two consecutive approximation is negligible.

This method of solving a system of equations is called the **Jacobi's iteration method.**

Note : In Jacobi's iteration method, we start with some initial approximation x_0, y_0, z_0 for the values of the unknowns. This initial value usually choosen as $x_0 = 0, y_0 = 0, z_0 = 0$. However, if an initial approximations are given, then we use these to start the iteration.

Example 1. Apply Jacobi iteration method to solve

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14$$

Solution : We write the given equations as

$$\begin{aligned} x &= \frac{12}{10} - \frac{y}{10} - \frac{z}{10} \\ y &= \frac{13}{10} - \frac{x}{5} - \frac{z}{10} \qquad \text{..... (1)} \\ z &= \frac{14}{10} - \frac{x}{5} - \frac{y}{5}. \end{aligned}$$

We start with initial approximation $x_0 = 0, y_0 = 0, z_0 = 0$. Putting these values in the equations, we get the first order approximation :

$$x_1 = \frac{12}{10} = 1.2, \quad y_1 = \frac{13}{10} = 1.3, \quad z_1 = \frac{14}{10} = 1.4$$

Putting these values on the right side of (1), we get the second order approximations.

$$x_2 = \frac{12}{10} - \frac{13}{100} - \frac{14}{100} = \frac{93}{100} = 0.93$$

$$y_2 = \frac{13}{10} - \frac{12}{50} - \frac{14}{100} = \frac{92}{100} = 0.92$$

$$z_2 = \frac{14}{10} - \frac{12}{50} - \frac{13}{50} = \frac{90}{100} = 0.90.$$

Substituting these values on the right side of (1), we get the third order approximations.

$$x_3 = \frac{12}{10} - \frac{92}{1000} - \frac{90}{1000} = \frac{1018}{1000} = 1.018$$

$$y_3 = \frac{13}{10} - \frac{93}{500} - \frac{90}{1000} = \frac{1024}{1000} = 1.024$$

$$z_3 = \frac{14}{10} - \frac{93}{500} - \frac{92}{500} = \frac{515}{500} = 1.030.$$

Again substituting these values on the right side of (1), we get the fourth approximations.

$$x_4 = \frac{12}{10} - \frac{1.024}{10} - \frac{1.030}{10} = 0.9946$$

$$y_4 = \frac{13}{10} - \frac{1.018}{5} - \frac{1.030}{10} = 0.9934$$

$$z_4 = \frac{14}{10} - \frac{1.018}{5} - \frac{1.024}{5} = 0.9916.$$

Again substituting these values in the right side of (1), we get the next approximations.

$$x_5 = \frac{12}{10} - \frac{0.9934}{10} - \frac{0.9916}{10} = 1.0015$$

$$y_5 = \frac{13}{10} - \frac{0.9946}{5} - \frac{0.9916}{10} = 1.0020$$

$$z_5 = \frac{14}{10} - \frac{0.9946}{5} - \frac{0.9934}{5} = 1.0024.$$

The values in the 4th and 5th iterations being almost same we can stop. Thus the solution is $x = 1,\ y = 1,\ z = 1$.

Example 2. Apply Jacobi iteration method to solve

$$5x - y = 9$$

$$-x + 5y - z = 4$$

$$y - 5z = 6$$

taking 1.8, 0.8 and – 1.2 as the initial approximation to the solution.

Solution : The given system can be written as

$$x = \frac{9}{5} + \frac{y}{5}$$

$$y = \frac{4}{5} + \frac{x}{5} + \frac{z}{5} \qquad \text{..... (1)}$$

$$z = -\frac{6}{5} + \frac{y}{5}.$$

The initial approximation is $x_0 = 1.8,\ y_0 = 0.8,\ z_0 = -1.2$. Putting these values in the right side of (1), we get the first approximation :

$$x_1 = \frac{9}{5} + \frac{0.8}{5} = 1.96$$

$$y_1 = \frac{4}{5} + \frac{1.8}{5} - \frac{1.2}{5} = 0.92$$

$$z_1 = -\frac{6}{5} + \frac{0.8}{5} = -1.04.$$

The second approximation is

$$x_2 = \frac{9}{5} + \frac{0.92}{5} = 1.984$$

$$y_2 = \frac{4}{5} + \frac{1.96}{5} - \frac{1.04}{5} = 0.984$$

$$z_2 = -\frac{6}{5} + \frac{0.92}{5} = -1.016.$$

The third approximation is

$$x_3 = \frac{9}{5} + \frac{0.8984}{5} = 1.9968$$

$$y_3 = \frac{4}{5} + \frac{1.984}{5} - \frac{1.016}{5} = 0.9936$$

$$z_3 = -\frac{6}{5} + \frac{0.984}{5} = -1.0032.$$

The fourth approximation is

$$x_4 = \frac{9}{5} + \frac{0.9936}{5} = 1.99872$$

$$y_4 = \frac{4}{5} + \frac{1.9968}{5} - \frac{1.0032}{5} = 0.99872$$

$$z_4 = -\frac{6}{5} + \frac{0.9936}{5} = -1.00128.$$

The fifth approximation is

$$x_5 = \frac{9}{5} + \frac{0.99872}{5} = 1.99974$$

$$y_5 = \frac{4}{5} + \frac{1.99872}{5} - \frac{1.00128}{5} = 0.999488$$

$$z_5 = -\frac{6}{5} + \frac{0.99872}{5} = -1.000256.$$

The sixth approximation is

$$x_6 = \frac{9}{5} + \frac{0.999488}{5} = 1.99989$$

$$y_6 = \frac{4}{5} + \frac{1.99974}{5} - \frac{1.000256}{5} = 0.99989$$

$$z_6 = -\frac{6}{5} + \frac{0.999488}{5} = -1.00010.$$

Clearly the values in the fifth and sixth iterates are very close to each other. Thus the solution is $x = 2, \ y = 1, \ z = -1$.

EXERCISES

1. Solve the following equations by Jacobi Iteration method.

   (a) $20x + y - 2z = 17; \quad 3x + 20y - z = -18; \ 2x - 3y + 20z = 25$

   (b) $27x + 6y - z = 85; \quad 6x + 15y + 2z = 72; \ x + y + 54z = 110$

   (c) $5x + 2y + z = 12; x + 4y + 2z = 15; x + 2y + 5z = 20$

   (d) $83x + 11y - 4z = 95; \ 7x + 52y + 13z = 104; \ 3x + 8y + 29z = 71.$

2. Solve the equations

   $5x - y + z = 10; \ x + 2y = 6; \ x + y + 5z = -1$

by Jacobi iteration method with (2, 3, 0) as initial approximation to the solution.

3. Solve the equations

$5x - y + 3z = 10;$ $3x + 6y = 18;$ $x + y + 5z = -10$

by Jacobi iteration method with (3, 0, −2) as initial approximation to the solution.

ANSWERS

1. (a) $x = 1, y = -1, z = 1$ (b) $x = 2.426, y = 3.573, z = 1.93$

(c) $x = 1.08, y = 1.95, z = 3.16$ (d) $x = 1.06, y = 1.37, z = 1.96$

2. $x = 2.556, y = 1.722, z = -1.055$ 3. $x = 4, y = 1, z = -3.$

1.12 Gauss – Seidel Iteration Method

This method is modification to Jacobi iteration method.

Consider the diagonally dominant equations,

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

As before we write this system as

$$x = \frac{1}{a_1}\left[d_1 - b_1 y - c_1 z\right] \quad \text{..... (1)}$$

$$y = \frac{1}{b_2}\left[d_2 - a_2 x - c_2 z\right] \quad \text{..... (2)}$$

$$z = \frac{1}{c_3}\left[d_3 - a_3 x - b_3 y\right] \quad \text{..... (3)}$$

Let the initial approximations of the solution be x_0, y_0, z_0 (each = 0). We shall put these values on the right side of the equation (1), we get,

$$x_1 = \frac{d_1}{a_1}$$

Now put $x = x_1 = \frac{d_1}{a_1}$ and $z = z_0$ in (2), we get

$$y_1 = \frac{1}{b_2}\left[d_2 - a_2 x_1\right] \quad [\text{since } z_0 = 0]$$

Now putting $x = x_1$ and $y = y_1$ in (3), we get

$$z_1 = \frac{1}{c_3}\left[d_3 - a_3 x_1 - b_3 y_1\right]$$

Now the first approximation of the solution is x_1, y_1, z_1. Continuing this way we get successive approximation of the solution and process may be stopped when the convergency to desired degree of accuracy is obtained.

Example 1. Using Gauss – Seidel iteration method solve the system.

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Solution : The given system can be written as

$$x = \frac{1}{27}(85 - 6y + z) \quad \text{..... (1)}$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad \text{..... (2)}$$

$$z = \frac{1}{54}(110 - x - y) \quad \text{..... (3)}$$

We start with initial approximation $x_0 = 0$, $y_0 = 0$, $z_0 = 0$. Putting these values in the equation (1), we get

$$x_1 = \frac{1}{27}(85) = 3.1481.$$

This is the first approximation of x.

Now put, $x_1 = 3.1481$ and $z = 0$ in (2), we get

$$y_1 = \frac{1}{15}[72 - 6(3.1481)] = 3.5407.$$

This is the first approximation of y.

Now put, $x_1 = 3.1481$ and $y_1 = 3.5407$ in (3), we get

$$z_1 = \frac{1}{54}[110 - 3.1481 - 3.5407] = 1.9131.$$

This is the first approximation of z.

Thus the first approximation of the solution is

$$x_1 = 3.1481,\ y_1 = 3.5407,\ z_1 = 1.9131.$$

Now putting these values in the equation (1), we get

$$x_2 = \frac{1}{27}[85 - 6(3.5407) + 1.9131] = 2.4321.$$

Now put, $x = 2.4321$ and $z = z_1 = 1.9131$ in (2), we get

$$y_2 = \frac{1}{15}[72 - 6(2.4321) - 2(1.9131)] = 3.5720$$

Now put, $x = x_2 = 2.4321$ and $y = y_2 = 3.5720$ in (3), we get

$$z_2 = \frac{1}{54}[110 - 2.4321 - 3.5720] = 1.9258$$

Thus the second approximation of the solution is

$$x_2 = 2.4321, \ y_2 = 3.5720, \ z_2 = 1.9258.$$

Now put, $y = y_2 = 3.5720$ and $z = z_2 = 1.9258$ in (1), we get

$$x_3 = \frac{1}{27}[85 - 6(3.5720) + 1.9258] = 2.4256.$$

Now put, $x = x_3 = 2.4256$ and $z = z_2 = 1.9258$ in (2), we get

$$y_3 = \frac{1}{15}[72 - 6(2.4256) - 2(1.9258)] = 3.5729.$$

Now put, $x = x_3 = 2.4256$ and $y = y_3 = 3.5729$ in (3), we get

$$z_3 = \frac{1}{54}[110 - 2.4256 - 3.5729] = 1.9259.$$

Thus the third approximation of the solution is

$$x_3 = 2.4256, \ y_2 = 3.5729, \ z_2 = 1.9259.$$

These values of x, y, z are sufficiently close to that of second approximations. Thus the solution is

$$x = 2.4256, \ y = 3.5729, \ z = 1.9259.$$

Example 2. Solve the following system of equations by Gauss – Seidel method.

$$\mathbf{28x + 4y - z = 32}$$
$$\mathbf{x + 3y + 10z = 24}$$
$$\mathbf{2x + 17y + 4z = 35}$$

Solution : The given system, as it is, is not diagonally dominant. However, by interchanging the second and the third equations, we can make it to be diagonally dominant equations. That is

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Now we have from these equations,

$$x = \frac{1}{28}[32 - 4y + z] \quad \text{..... (1)}$$

$$y = \frac{1}{17}[35 - 2x - 4z] \quad \text{..... (2)}$$

$$z = \frac{1}{28}[24 - x - 3y] \quad \text{..... (3)}$$

We start with initial approximation, $x_0 = 0, \ y_0 = 0, \ z_0 = 0.$

First iteration.

Put $y = 0, \ z = 0$ in (1); $x = \frac{1}{28}(32) = 1.1429$

Put $x = 1.1429, \ z = 0$ in (2); $y = \frac{1}{17}(35 - 2.2858) = 1.9244$

Put $x = 1.1429, \ y = 1.9244$ in (3); $z = \frac{1}{10}(24 - 1.1429 - 5.7732) = 1.7084$

Thus the first approximation of the solution is

$$x = 1.1429, \ y = 1.9244, \ z = 1.7084$$

Second iteration.

Put $y = 1.9244, \ z = 1.7084$ in (1); $x = \frac{1}{28}(32 - 7.6976 + 1.7084) = 0.9290$

Put $x = 0.9290, \ z = 1.7084$ in (2); $y = \frac{1}{17}(35 - 1.858 - 6.8336) = 1.5476$

Put $x = 0.9290, \ y = 1.5476$ in (3); $z = \frac{1}{10}(24 - 0.929 - 4.6428) = 1.8428$

Thus the second approximation of the solution is

$$x = 0.929, \ y = 1.5476, \ z = 1.8428$$

Third iteration.

Put $y = 1.5476, \ z = 1.8428$ in (1); $x = \frac{1}{28}(32 - 6.1904 + 1.8428) = 0.9876$

Put $x = 0.9876, \ z = 1.8428$ in (2); $y = \frac{1}{17}(35 - 1.9752 - 7.3712) = 1.5090$

Put $x = 0.9876, \ y = 1.5090$ in (3); $z = \frac{1}{10}(24 - 0.9876 - 4.527) = 1.8485$

Thus the third approximation of the solution is

$$x = 0.9876, \ y = 1.5090, \ z = 1.8485$$

Fourth iteration.

Put $y = 1.5090, \ z = 1.8485$ in (1); $x = \frac{1}{28}(32 - 6.036 + 1.885) = 0.9933$

Put $x = 0.9933, \ z = 1.8485$ in (2); $y = \frac{1}{17}(35 - 1.9866 - 7.394) = 1.5070$

Put $x = 0.9933, \ y = 1.5070$ in (3); $z = \frac{1}{10}(24 - 0.9266 - 4.521) = 1.8486$

Thus the fourth approximation of the solution is

$$x = 0.9933, \ y = 1.5070, \ z = 1.8486$$

Fifth iteration.

Put $y = 1.5070, \ z = 1.8486$ in (1); $x = \frac{1}{28}(32 - 6.028 + 1.8486) = 0.9936$

Put $x = 0.9936,\ z = 1.8486$ in (2); $y = \frac{1}{17}(35 - 1.9872 - 7.3944) = 1.5070$

Put $x = 0.9936,\ y = 1.5070$ in (3); $z = \frac{1}{10}(24 - 0.9936 - 4.521) = 1.8485$

Thus the fifth approximation of the solution is

$$x = 0.9936,\ y = 1.5070,\ z = 1.8485$$

The fourth and fifth iteration give almost the same values. Thus the solution is

$$x = 0.9936,\ y = 1.5070,\ z = 1.8485$$

EXERCISES

1. Use Gauss – Seidel iteration method to solve the following equations.

   (a) $10x + y + z = 12;\quad 2x + 10y + z = 13;\quad 2x + 2y + 10z = 14$

   (b) $5x - y = 9;\ x - 5y + z = -4;\quad y - 5z = 6$

   (c) $25x + 2y + 2z = 69; 2x + 10y + z = 63; x + y + z = 43$

   (d) $9.37x + 3.04y - 2.44z = 9.23; 3.04x + 6.18y + 1.22z = 8.2;$
   $-2.44x + 1.22y + 8.844z = 3.93.$

   (e) $4x + y + 2z = 4; 3x + 5y + z = 7; x + y + 3z = 3$

   (f) $20x + 2y + 6z = 28; x + 20y + 9z = -23;\quad 2x - 7y - 20z = -57.$

ANSWERS

1. (a) $x = 1, y = -1, z = 1$

   (b) $x = 2,\ y = 1,\ z = -1$

   (c) $x = 0.9953, y = 2.116, z = 39.8931$

   (d) $x = 0.8841, y = 0.7773, z = 0.5810$

   (e) $x = 0.52, y = 0.992, z = 0.496$ (f) $x = 0.5149, y = -2.94514, z = 3.9323.$

1.13 Finite Differences

Let $y = f(x)$ be a function, which assumes the values $f(a)$, $f(a + h)$, $f(a + 2h)$, ... corresponding to the values of x, namely a, $a + h$, $a + 2h$, ... The each value of x is called **argument** and the corresponding values of y is known as **entry**.

The first forward difference of $y = f(x)$ is defined as

$$\Delta f(x) = f(x + h) - f(x)$$

The second forward difference of $y = f(x)$ is defined as

$$\Delta^2 f(x) = \Delta f(x + h) - \Delta f(x)$$

The general nth forward difference of $y = f(x)$ is defined as

$$\Delta^n f(x) = \Delta^{n-1} f(x + h) - \Delta^{n-1} f(x)$$

Here Δ is the forward difference operator.

If the arguments are denoted by $x_0, x_1, x_2,$ ..... and the corresponding entries by $y_0, y_1, y_2,$ ..... , then the first, second and higher order forward differences are given by

$$\Delta y_r = y_{r+1} - y_r$$

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

and $$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r.$$

The differences can be represented in the form of a table called **forward difference table.**

value of x	value of y	1st Diff	2nd Diff	3rd Diff	4th Diff	5th Diff
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

Here y_0 the first entry is called the **leading entry** (term) and Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ and $\Delta^5 y_0$ are called the leading differences. The quantity h is called the **interval of differencing.**

Note : Any higher order forward differences can be expressed in terms of entries.

Now, $$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$\Rightarrow$ $$\Delta^2 y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$\Rightarrow$ $$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

Again, $$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$\Rightarrow$ $$\Delta^3 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$

$\Rightarrow$ $$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0.$$

Similarly, $$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 43y_1 + y_0$$

In general,

$$\Delta^n y_0 = y_n - nC_1 y_{n-1} + nC_2 y_{n-2} + \dots\dots\dots + (-1)^n y_0$$

Observe that the coefficients on the right side are binomial coefficients.

In general,

$$\Delta^n y_r = y_{n+r} - nC_1 y_{n+r-1} + nC_2 y_{n+r-2} + \dots\dots\dots + (-1)^n y_r$$

Example 1. Construct the forward difference table, given that

x	5	10	15	20	25	30
y	9962	9848	9659	9397	9063	8660

and write down the values of $\Delta^2 y_{10}$, $\Delta^3 y_5$, $\Delta^4 y_5$

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962				
		-114			
10	9848		-75		
		-189		2	
15	9659		-73		-1
		-262		1	
20	9397		-72		2
		-334		3	
25	9063		-69		
		-403			
30	8660				

Now $\Delta^2 y_{10} = -73$ which is the second element of the column $\Delta^2 y$.

$\Delta^3 y_5 = 2$ which is the first element of the column $\Delta^3 y$. Similarly $\Delta^4 y_5 = -1$.

Example 2. Construct the forward difference table of the polynomial $f(x) = x^2 + x + 1$ for values $x = 0(1)4$ (i.e. values of $f(x)$ from 0 to 4 with interval of difference 1)

Solution : $f(x) = x^2 + x + 1$ and $x = 0, 1, 2, 3, 4$. That is x varies from 0 to 4 with 1 as interval of difference.

Thus we have,

x	0	1	2	3	4
$f(x)$	1	3	7	13	21

The difference table is

x	$f(x)$	Δ	Δ^2	Δ^3
0	1			
		2		
1	3		2	
		4		1
2	7		2	
		6		1
3	13		2	
		8		
4	21			

Example 3. Construct the forward difference table of the polynomial $f(x) = x^3 + x^2 - 2x + 1$ for values $x = 0(1)\,5$. Find the value of the polynomial at $x = 6$, by extending the table.

Solution : $f(x) = x^3 + x^2 - 2x + 1$ and $x = 0, 1, 2, 3, 4, 5$. Thus we have,

x	0	1	2	3	4	5
$y = f(x)$	1	1	9	31	73	141

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		0		
1	1		8	
		8		6
2	9		14	
		22		6
3	31		20	
		42		6
4	73		26	
		68		**6**
5	141		**26 + 6 = 32**	
		68 + 32 = 100		
6	**141 + 100 = 241**			

Now from the above table we have $\Delta^3 y_3 = 6$

Now, $\Delta^3 y_3 = \Delta^2 y_4 - \Delta^2 y_3$

$\Rightarrow \quad \Delta^2 y_4 = \Delta^3 y_3 + \Delta^2 y_3 = 26 + 6 = 32.$

Now, $\Delta^2 y_4 = \Delta y_5 - \Delta y_4$

$\Rightarrow$ $\Delta y_5 = \Delta^2 y_4 + \Delta y_4 = 32 + 68 = 100.$

Now, $\Delta y_5 = y_6 - y_5$

$\Rightarrow$ $y_6 = \Delta y_5 + y_5 = 141 + 100 = 241.$

Thus $f(6) = 24$

Now putting $x = 6$ in $f(x) = x^3 + x^2 - 2x + 1$, we get

$$f(6) = 6^3 + 6^2 - 2(6) + 1 = 196 + 36 - 11 = 241$$

EXERCISES

1. Construct the forward difference table from the following data and find the value of $\Delta^3 f(2)$.

x	0	1	2	3	4
$f(x)$	1.0	1.5	2.2	3.1	4.6

2. Construct the forward difference table from the following data

(a)

x	0.1	0.2	0.3	0.4
$f(x)$	15.21	15.44	15.69	15.96

(b)

x	0	2	4	6	8
$f(x)$	40	51.68	67.04	86.56	110.72

1.14 Backward Difference Operator ∇

The forward difference of $f(x)$ is defined as

$$\Delta f(x) = f(x + h) - f(x)$$

Here $\Delta f(x)$ is expressed in terms of $f(x)$ and the functional value one step forward – i.e. $f(x + h)$.

Now we define backward difference of $f(x)$ by expressing the difference using $f(x)$ and the functional value one step backwards.

The backward difference is defined as

$$\nabla f(x) = f(x) - f(x - h)$$

Here ∇ is called the **backward difference operator.**

Similarly, $\nabla^2 f(x) = \nabla f(x) - \nabla f(x - h)$

In general, $\nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(x - h)$

Example 1. Construct the backward difference table for

x	10	20	30	40	50
y	1	1.3010	1.4771	1.6021	1.6990

Solution :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1				
		0.3010			
20	1.3010		−0.1249		
		0.1761		0.0738	
30	1.4771		−0.0511		−0.0508
		0.1250		0.0230	
40	1.6021		−0.0281		
		0.0969			
50	1.6990				

Here $\nabla^2 y_{40} = -0.0511$, $\quad \nabla^4 y_{50} = -0.0508$.

1.15 The Operator E – Shift Operator

The shift operator E is defined by

$$Ef(x) = f(x+h) \qquad \text{or} \qquad Ey_x = y_{x+h}$$

Similarly, $\quad E^2 f(x) = E[Ef(x)] = Ef(x+h) = f(x+2h)$

$E^3 f(x) = f(x+3h)$ and so on.

In general, $\quad E^n f(x) = f(x+nh)$.

The operator E^{-1} is defined by

$$E^{-1}[Ef(x)] = f(x)$$

or $\quad E^{-n}[E^n f(x)] = f(x)$

The following properties can be verified easily.

1. $E[af(x) + bg(x)] = aEf(x) + bEg(x)$
2. $E^m[E^n f(x)] = E^{m+n} f(x)$

1.16 Relation between the operators Δ, E, ∇, $D = \frac{d}{dx}$.

In this section we shall see few relation connecting Δ, E, ∇, and $D = \frac{d}{dx}$.

1. **The operators E and Δ are commutative.**

$$\begin{aligned} E[\Delta f(x)] &= E[f(x+h) - f(x)] \\ &= Ef(x+h) - Ef(x) \\ &= f(x+2h) - f(x+h) \\ &= \Delta f(x+2h) = \Delta[Ef(x)] \end{aligned}$$

Thus, $\quad E\Delta = \Delta E$

2. $\mathbf{E = 1 + \Delta}$ **or** $\mathbf{\Delta = E - 1}$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Rightarrow \quad f(x+h) = f(x) + \Delta f(x)$$

$$\Rightarrow \quad Ef(x) = (1+\Delta)f(x)$$

Thus $\quad E = 1 + \Delta \quad$ or $\quad \Delta = E - 1.$

3. $\mathbf{\nabla = 1 - E^{-1}}$ **and** $\mathbf{\nabla = \Delta E^{-1}}$

Consider, $\quad \nabla f(x) = f(x) - f(x-h)$

$$\Rightarrow \quad \nabla f(x) = \Delta f(x) - E^{-1} f(x)$$

$$\Rightarrow \quad \nabla f(x) = (1 - E^{-1}) f(x)$$

$$\Rightarrow \quad \nabla = 1 - E^{-1}$$

Again Consider, $\quad \nabla f(x) = f(x) - f(x-h)$

$$\Rightarrow \quad \nabla f(x) = \Delta f(x-h)$$

$$\Rightarrow \quad \nabla f(x) = \Delta E^{-1} f(x) \quad \Rightarrow \quad \nabla = \Delta E^{-1}$$

4. $\mathbf{E = e^{hD}}$ **or** $\mathbf{hD = \log_e E}$

We have by Taylor's series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \ldots..$$

$$\Rightarrow \quad Ef(x) = f(x) + \frac{h}{1!}Df(x) + \frac{h^2}{2!}D^2 f(x) + \ldots..$$

$$\Rightarrow \quad E = 1 + \frac{h}{1!}D + \frac{h^2}{2!}D^2 + \ldots..$$

$$\Rightarrow \quad E = e^{hD} \text{ or } \quad hD = \log_e E.$$

5. $$\mathbf{\Delta^n f(x) = f(x+nh) - \frac{n}{1!}f(x+\overline{n-1}\,h) + \frac{n(n-1)}{2!}f(x+\overline{n-2}\,h) - \ldots\ldots\ldots + (-1)^n f(x)}$$

Consider, $\quad \Delta^n f(x) = (E-1)^n f(x)$

$$= \left[E^n - \frac{n}{1!}E^{n-1} + \frac{n(n-1)}{2!}E^{n-2} - \ldots.. + (-1)^n . 1\right] f(x)$$

$$= E^n f(x) - \frac{n}{1!}E^{n-1} f(x) + \frac{n(n-1)}{2!}E^{n-2} f(x) - \ldots.. + (-1)^n . 1 f(x)$$

$$= f(x+nh) - \frac{n}{1!}f(x+\overline{n-1}\,h) + \frac{n(n-1)}{2!}f(x+\overline{n-2}\,h) - \ldots.. + (-1)^n . 1 f(x)$$

which is the required result. This expresses the n^{th} difference of a function in terms of successive entries.

6. $f(x + nh) = f(x) - \frac{n}{1!}\Delta f(x)$

$$+ \frac{n(n-1)}{2!}\Delta^2 f(x) + \text{..........} + \Delta^n f(x)$$

Consider, $f(x + nh) = E^n f(x)$

$$= (1 + \Delta)^n f(x)$$

$$= \left[1 + \frac{n}{1!}\Delta + \frac{n(n-1)}{2!}\Delta^2 + \text{..........} + \Delta^n\right] f(x)$$

$$f(x + nh) = f(x) - \frac{n}{1!}\Delta f(x) + \frac{n(n-1)}{2!}\Delta^2 f(x) + \text{.....} + \Delta^n f(x)$$

which is the required result. This expresses the function $f(x + nh)$ in terms of $f(x)$ and its successive differences.

1.17 Interpolation

Let $y = f(x)$ be a function and the corresponding values of y, i.e. $f(x)$ for a set of values of x, say $a, a + h, a + 2h, \text{..........}, a + nh$ are given by $y_a, y_{a+h}, y_{a+2h}, \text{..........}, y_{a+nh}$. Interpolation is a process of finding the value of y lying between two adjescent values of x. Finding the value of $f(x)$ corresponding to a value of x just out side the range of x is called **extrapolation.**

Thus interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable.

We derive two interpolation formulae known as **Newton – Gregory forward interpolation** formula and **Newton – Gregory backward interpolation** formula. The first one is used for interpolating the values near the beginning of a set of tabulated values and extrapolating values of y a little backward of y_0; the second one is for interpolating the values of y near the end of a set of tabulated values and extrapolating values of y little ahead of y_n.

The assumption made to derive these formulae is that there is a polynomial function whose graph is very nearly the same as that of true curve. Validity of this assumption of replacing $f(x)$ by a polynomial function is due to an important theorem of **Weierstrass.** This theorem states that "**if $f(x)$ is continuous between two values $x = x_0$ and $x = x_n$, then it can be replaced by a polynomial of suitable degree in that interval with as small an error as we please.**"

1. Newton – Gregory interpolation formula

Newton – Gregory forward interpolation formula

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \text{..........}$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \text{............}$ of x. Let it is required to find the value of $f(x)$ for $x = x_0 + ph$, where p is any real number.

Now we have for any real number p

$$E^p f(x) = f(x + ph)$$

Now, $y_x = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0$

$$\Rightarrow \quad y_x = \left\{ 1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\ldots \right\} y_0$$

$$\Rightarrow \quad y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

This formula is known as **Newton – Gregory forward interpolation** formula.

Newton – Gregory backward interpolation formula

Let the funct ion $y = f(x)$ take the values $y_0, y_1, y_2, \ldots\ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots\ldots$ of x. Let it is required to find the value of $f(x)$ for $x = x_n + ph$, where p is any real number.

We have

$$E^{-1} = 1 - \nabla, \quad \text{i.e.} \quad E = (1 - \nabla)^{-1}$$

Now, $y_x = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n$

$$\Rightarrow \quad y_x = \left\{ 1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\ldots \right\} y_n$$

$$\Rightarrow \quad y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

This formula is known as **Newton – Gregory backward interpolation** formula.

Example 1. Given that sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192, sin 60° = 0.8660, find sin 52°, using Newton – Gregory forward interpolation formula.

Solution : The difference table is

x°	$10^4 y$	$10^4\Delta$	$10^4\Delta^2$	$10^4\Delta^3$
45	7071			
		589		
50	7660		−57	
		532		−7
55	8192		−64	
		468		
60	8660			

We shall take,

$$x_0 = 45, \; h = 5 \text{ and } y_0 = 7071$$

Here $x = 52$ and also

$$x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h} = \frac{52 - 45}{5} = 1.4$$

Newton – Gregory forward interpolation formula is